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1990 J. Phys. A: Math. Gen. 23 3409

(http://iopscience.iop.org/0305-4470/23/15/016)

J. Phys. A: Math. Gen. **23** (1990) 3409-3420. Printed in the UK

A new physical basis for the irreducible representations of the orthogonal group SO(5) in the quasi-spin formalism

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Received 20 October 1990

Abstract. New physical state vectors for the IRs of the group SO(5) are constructed in an analytical form. The state vectors are labelled by the quantum numbers of the isospin, its third component, by the number of particles and by a fourth properly defined quantum number. The previously constructed bases have been compared with the new construction from the same point of view. Two examples illustrate the new construction.

1. Introduction

The group of orthogonal transformations SO(5) in five-dimensional abstract space, locally isomorphic with the symplectic group Sp(4), has found many applications in physics and particularly in nuclear physics. Symmetric representations of the group SO(5) described the octupole vibrations of nuclei and it is also a symmetry of the interacting boson model in its vibrational limit (Arima and Iachello 1976). In the seniority scheme the group SO(5) has been applied to the classification of neutron-proton states (Helmers 1961) and to the diagonalisation of the pairing Hamiltonian (Flowers and Szpikowski 1964, Ichimura 1964, Hecht 1965, Parikh 1965). In the physical applications, the bases of the irreducible representations (IRs) of the group are needed for the construction of matrix elements of the important physical operators. For a given IR labelled by two quantum numbers (see for example Szpikowski 1966) one needs four further quantum numbers to factorise the states within a given IR. The complete classification of the state vectors was provided by the reduction of the group SO(5) (Hecht 1965):

$$\mathrm{SO}(5) \supset \mathrm{SU}_n(2) \times \mathrm{SU}_p(2) \tag{1}$$

where the groups $\mathrm{SU}_n(2)$ and $\mathrm{SU}_p(2)$ are the quasi-spin groups separately for neutrons and protons. However, the four quasi-spin neutron and proton numbers are not good physical numbers. A more significant reduction is given by

$$\mathrm{SO}(5) \supset \mathrm{SU}_T(2) \times \mathrm{U}(1) \tag{2}$$

where the group $\mathrm{SU}_T(2)$ is the ordinary isospin group and U(1) is a particle-generator transformation group (Flowers and Szpikowski 1964). The reduction (2) provides three physical quantum numbers: the isospin T, its third component M_T and the number

§ Work partially supported by Polish Ministry of National Education, Project CPBP 01.06.
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0305-4470/90/153409+12$03.50

of particles N. The missing fourth quantum number represents the known crucial problem in spite of the fact that the fourth commuting operator had been constructed a long time ago (Flowers and Szpikowski 1965, Hecht 1967).

For the simplest IRs the construction of the state vectors have been described in detail (Szpikowski 1966, Hecht 1967, Hemenger and Hecht 1970). The general basis construction for a given IR is described by Ahmed and Sharp (1970). Another solution of the same problem has been found by Smirnov and Tolstoy (1973) and the analysis of the solution has been discussed by Ališauskas (1983, 1984). The constructed state vectors are analytical, complete but not orthogonal ones. Recently, Hecht and Elliott (1985) have discussed the same problem within the coherent-state method; however their procedure leads to a non-analytical solution.

In this paper we propose a new construction of the state vectors which provide the basis for a given IR of the group SO(5). Construction of the state vectors is given in section 2 and in section 3 we prove that the basis so constructed is a complete one (even overcomplete). In that section we also give a rule for choosing the exact number of linearly independent basis vectors. Also in section 3 we have compared our new construction with the known constructed bases. The discussion of similarities and differences of the bases is given in section 4. Finally, in section 5, we have discussed some applications of the constructed basis to physical problems, namely to the fourth commuting operator and to the problem of a pairing interaction.

2. The construction of the basis for a given IR of the group SO(5)

We adopt the following definition of the generators of SO(5) transformations in the quasi-spin space in terms of the creation $a^{\dagger}_{jmm_t}$ and annihilation a_{jmm_t} nucleon operators (we adopt $m_t = \frac{1}{2}$ for a neutron) on the j level:

$$
\begin{aligned}
&A^{\dagger}(M_T) = \frac{1}{2}\sum_{mm_t}(-1)^{j-m}(\tfrac{1}{2}m_t\tfrac{1}{2}m_{t'}|1M_T)a^{\dagger}_{jmm_t}a^{\dagger}_{j-mm'_t}\\
&A(M_T) = [A^{\dagger}(M_T)]^{\dagger}\\
&T_{\pm} = \sum_m a^{\dagger}_{jm\pm 1/2}a_{jm\mp 1/2} \qquad (3)\\
&T_0 = \frac{1}{2}\sum_m (a^{\dagger}_{jm+1/2}a_{jm+1/2} - a^{\dagger}_{jm-1/2}a_{jm-1/2})\\
&N_0 = \frac{1}{2}\sum_{mm_t} a^{\dagger}_{jmm_t}a_{jmm_t} - (j+\tfrac{1}{2}).
\end{aligned}
$$

The first two operators create (annihilate) the pair of nucleons coupled to a zero angular momentum; the next three T operators are the isospin operators and the last one, N_0, is the number of particles operator modified by a factor $\frac{1}{2}$ and a constant $-(j+\frac{1}{2})$. The quasi-spin operators, separately for neutrons and protons, read

$$
\begin{aligned}
&S^{+}_n = A^{\dagger}(1) \qquad S^{-}_n = A(1) \qquad S^{0}_n = \tfrac{1}{2}(N_0 + T_0)\\
&S^{+}_p = A^{\dagger}(-1) \qquad S^{-}_p = A(-1) \qquad S^{0}_p = \tfrac{1}{2}(N_0 - T_0).
\end{aligned} \qquad (4)
$$

The weight operators are N_0 and T_0. The maximum weight (ω_1, ω_2) can be expressed by the physical numbers seniority v and reduced isotopic spin t (Flowers and

Szpikowski 1964)

$$\omega_1 = (j+\tfrac{1}{2}) - \frac{v}{2} \qquad \omega_2 = t. \tag{5}$$

The maximum weight in this work will be labelled by (ω, t) where $\omega \equiv \omega_1$.

We begin the construction with the state vector $|vt\rangle$ where

$$|vt\rangle \equiv |(\omega t) N = v;\ T = M_T = t\rangle \tag{6}$$

which is a unique state, i.e. the fourth quantum number, β, takes on only a unique value in the state (6). It is a state of the non-paired particles only, with maximum value of the isospin third component. The state (6) is not explicitly constructed but it has specific properties under the generator transformations. For example

$$A(M_T)|vt\rangle = T_+|vt\rangle = 0. \tag{7}$$

The general state construction has been done in several steps as follows.

(i) The maximum number of neutron pairs each coupled to $J = 0$ is created in the state (6):

$$A^+(1)^{\omega - t}|vt\rangle = |(\omega t);\ N' = v + 2(\omega - t);\ T' = M'_T = \omega\rangle. \tag{8}$$

(ii) From the state (8) with N' and $T' = \omega$ we generate the state of a given *a priori* N and T within a given IR (ω, t) by means of the operator (Szpikowski and Góźdź 1980)

$$\hat{0}(T'lT) = \sum_{r=0}^{\omega - T} \frac{(-1)^r}{r!(2T+r+1)!}\, T_-^{r+T-M_T} T_+^{v+T-\omega+l} \mathcal{T}^{(l)}_{-l} \tag{9}$$

where $\mathcal{T}^{(l)}$ is the tensor operator of a rank l in the isospin space. The operator $\mathcal{T}^{(l)}$ has to be constructed from the generators (3). In the case under consideration we adopt the following form of $\mathcal{T}^{(l)}$

$$\mathcal{T}^{(l)}_{-l} = (-1)^{l-\beta} A^+(-1)^\beta A(1)^{l-\beta} \tag{10}$$

where β is, for the moment, a free index. The rank l has been chosen in such a way as to obtain a prescribed number of particles N, i.e.

$$l = (\omega - t) - \tfrac{1}{2}(N - v) + 2\beta. \tag{11}$$

If we apply the operators (9), (10) to the state (8) and commute the annihilation operator $A(1)$ to the right we get, up to a constant factor,

$$|(\omega t) N \beta T M_T\rangle = \sum_{r=0}^{\omega - T} \frac{(-1)^r}{r!(2T+r+1)!}\, T_-^{r+T-M_T} T_+^{r+T-K} A^+(-1)^\beta A^+(1)^{p-\beta}|vt\rangle \tag{12}$$

where $p = \frac{1}{2}(N - v)$ and $K = t + p - 2\beta$.

(iii) It can be seen that the state vectors (12) transform to the form (up to a constant factor):

$$|(\omega t) N \beta T M_T\rangle = P^T_{M_T K} A^+(-1)^\beta A^+(1)^{p-\beta}|vt\rangle \tag{13}$$

where the operator

$$P^T_{M_T K} = (2T+1) \sqrt{\frac{(T+M_T)!(T+K)!}{(T-M_T)!(T-K)!}} \sum_r \frac{(-1)^r}{r!(2T+r+1)!}\, T_-^{r+T-M_T} T_+^{r+T-K} \tag{14}$$

is a projection operator for an isospin group (Löwdin 1964, Shapiro 1965).

Straightforward inspection shows that the states ('internal states')

$$A^+(-1)^\beta A^+(1)^{p-\beta}|vt\rangle \tag{15}$$

are the eigenstates of the quasispin operators (4) (squares of quasispins and their third components) and the eigenvalues read

$$\begin{aligned} S_n &= \tfrac{1}{2}(\omega - t) \qquad & S_n^0 &= p - \beta - \tfrac{1}{2}(\omega - t) \\ S_p &= \tfrac{1}{2}(\omega + t) \qquad & S_p^0 &= \beta - \tfrac{1}{2}(\omega - t). \end{aligned} \tag{16}$$

We can say, by (13) and (15), that the state vectors can be projected from 'internal states' (15) with the help of the projection operator (14). We can extract from the construction (13) the possible values of the parameter β for a given IR (ω, t), namely

$$\max(0, p + t - \omega) \leqslant \beta \leqslant \min(p, t + \omega). \tag{17}$$

Hence, we have completed the construction of the general basis for a given IR of the group SO(5). In the next section we give the completeness proof. In fact, the basis is overcompleted and, hence, we also give a rule for obtaining an exact number of state vectors.

3. The completeness proof

Due to the special choice of the 'internal states' (15), the completeness proof can be done in a similar way as in Elliott's proof (1958) for the basis functions in the reduction $SU(3) \supset SO(3)$.

Suppose the state vectors (13) do not form a complete set. Hence, in a space of the IR basis there exists a non-zero vector $|\tilde{N}\tilde{T}\tilde{M}_T\rangle$ orthogonal to vectors (13). Hence

$$\langle \tilde{N}\tilde{T}\tilde{M}_T | P^T_{M_T K} A^+(-1)^\beta A^+(1)^{p-\beta}|vt\rangle = 0. \tag{18}$$

If the state vector (13) has one (or more) different values $N \neq \tilde{N}$; $T \neq \tilde{T}$ or $M_T \neq \tilde{M}_T$ then (18) is necessarily fulfilled. Hence, let us take $N = \tilde{N}$; $T = \tilde{T}$; $M_T = \tilde{M}_T$. The action in (18) of the operator $P^T_{M_T K}$ to the left gives

$$\langle \tilde{N}\tilde{T}K | A^+(-1)^\beta A^+(1)^{p-\beta}|vt\rangle = 0. \tag{19}$$

If $K \neq \tilde{M}_T$ then (19) also holds. Hence, the vector $|\tilde{N}\tilde{T}\tilde{M}_T\rangle$ is also orthogonal to the 'internal states' (15). Let us then consider any polynomial G of the SO(5) generators in which we put the generators in the order (from left to right): $T_\pm$; $A^+(0)$; $A(0)$ and the rest of the generators in any order. The rest of the generators while acting on 'internal states' give also 'internal states'. Then the action of the operators $A^+(0)$ and $A(0)$ can be equivalently transformed to the action of T_+ and T_- operators according to formulae:

$$\begin{aligned} A^+(0)\{A^+(-1)^\beta A^+(1)^{p-\beta}|vt\rangle\} &= \frac{1}{\sqrt{2}(p+1)} T_+ A^+(-1)^{\beta+1} A^+(1)^{p-\beta}|vt\rangle \\ A(0)\{A^+(-1)^\beta A^+(1)^{p-\beta}|vt\rangle\} &= -\frac{p-\beta}{\sqrt{2}} T_- A^+(-1)^{\beta-1} A^+(1)^{p-\beta}|vt\rangle. \end{aligned} \tag{20}$$

Hence, the polynomial G while acting on the 'internal state' gives

$$GA^+(-1)^\beta A^+(1)^{p-\beta}|vt\rangle = \sum_{p'\beta'mn} a_{p'\beta'mn} T_+^m T_-^n A^+(-1)^{\beta'} A^+(1)^{p'-\beta'}|vt\rangle \tag{21}$$

where a are expansion coefficients.

Then we get

$$\langle \tilde{N}\tilde{T}\tilde{M}_T | GA^+(-1)^{\beta}A^+(1)^{p-\beta}|vt\rangle$$
$$= \sum_{p'\beta' mn M'_T} a_{p'\beta' mn} b_{M_T M'_T} \langle \tilde{N}\tilde{T}M'_T | A^+(-1)^{\beta'}A^+(1)^{p'-\beta'}|vt\rangle. \tag{22}$$

The right-hand side of (22) is zero from (19). Hence, the left-hand side of (22) is also zero. But the arbitrary polynomial G of generators (3) can generate any of the basis vectors for a given IR. Hence, the vector $|\tilde{N}\tilde{T}\tilde{M}_T\rangle$, from the space of basis vectors of a given IR, giving the scalar product with any of the vectors equal to zero must be itself a zero vector. In other words, such a vector does not exist and the basis (13) forms a complete set.

Now we proceed to choose the linearly independent set of states (13) which form a complete, but not overcomplete, basis of a given IR of the group SO(5). Let us consider only those vectors (13) which fulfil the rules:

(i) for $T \leqslant t$ we take all of the states with allowed β given by (17);

(ii) for a given $T > t$ we choose the states (13) only with β given by the relation

$$|K| \leqslant T \leqslant \min(K+\beta, K+2t). \tag{23}$$

If we calculate the number of states (13) bounded by (23) we get the number

$$\tfrac{1}{6}(2\omega+3)(2t+1)(\omega+t+2)(\omega-t+1) \tag{24}$$

which is exactly the number of the basis states for a given irreducible representation.

It can be also shown that the states for a given T

$$T > \min(K+\beta, K+2t) \tag{25}$$

are the linear combination of the states with the same T but with K' and β' allowed by the relation

$$T \leqslant \min(K'+\beta', K'+2t). \tag{26}$$

For example, if T fulfils (25), say $T > K+\beta$, then

$$|(\omega t)N\beta TM_T\rangle = \sum_{\beta' \geqslant T-K'} C'_{\beta} |(\omega t)N\beta' TM_T\rangle. \tag{27}$$

The basis vectors of the complete (exactly) set are not, however, orthogonal. The calculated scalar product of two vectors which differ by the β number only reads

$$\langle (\omega t)N\beta' TM_T | (\omega t)N\beta TM_T\rangle$$
$$= (2T+1)\sqrt{\frac{(T+K')!(T+K)!}{(T-K')!(T-K)!}}\,\frac{(\omega+t+1)!(2t)!(2\omega+2)!(\omega-t)!\beta'!\beta!}{(\omega+t-\beta')!(\omega+t-\beta)!}$$
$$\times \sum_{r,i,l} \frac{(-1)^{2l+\beta'+\beta+T-2t-\omega-p}(2i+1)(2l+1)(T-K'+r)!(T-K+r)!}{r!(2T+r+1)!\nabla^2(\frac{1}{2}(\omega+t)-l, \frac{1}{2}(\omega-t), i)E^2(\frac{1}{2}(\omega+t)+l, i, \frac{1}{2}(\omega-t))}$$
$$\times \frac{(i+\frac{1}{2}(T-\omega+p+r))!(l+\frac{1}{2}(T+\omega-p+r))!}{(i-\frac{1}{2}(T-\omega+p+r))!(l-\frac{1}{2}(T+\omega-p+r))!}$$
$$\times \frac{(l-\beta'+t-\frac{1}{2}(T-\omega-p+r))!(l-\beta+t-\frac{1}{2}(T-\omega-p+r))!}{(l+\beta'-t+\frac{1}{2}(T-\omega-p+r))!(l+\beta-t+\frac{1}{2}(T-\omega-p+r))!} \tag{28}$$

where

$$\nabla(a, b, c) \equiv [(a+b-c)!(a-b+c)!(a+b+c+1)!/(b+c-a)]^{1/2}$$
$$E(a, b, c) \equiv [(a-b-c)!(a-b+c+1)!(a+b-c+1)!(a+b+c+2)!]^{1/2}.$$

In the above result we have used the eigenvalues of the quasi-spin operators in the 'internal state' (16) as well as the matrix elements of the $T_{\pm}$ operators given by Ališauskas and Jucys (1971).

4. Unified description of the different constructions of the basis vectors

The construction of the basis given in this paper as well as the constructions of Ahmed and Sharp (1970) and Smirnov and Tolstoy (1973) can be shown to rely on the same projection method with the differently defined 'internal states'. The differences of the definition of 'internal states' in the three constructions are given in figure 1, where the 'initial states' are the 'corner states' on the (N_0, T) diagram. Although Ahmed and Sharp took the nine basis vectors of two fundamental representations of the group SO(5), their vectors can be equivalently constructed starting with the highest weight state

$$|HW\rangle = |(\omega t)N = 2\omega + v;\ T = M_T = t\rangle. \tag{29}$$

The construction of Smirnov and Tolstoy began with the state

$$|HW\rangle' = |(\omega t)N = 2t + v;\ T = M_T = \omega\rangle \tag{30}$$

that is the state of a highest weight if for the first weight operator T_0 is taken but not N_0. Let us repeat that our construction is based on the initial state (6)

$$|vt\rangle = |(\omega t)N = v;\ T = M_T = t\rangle. \tag{6}$$

The initial states (29), (30), (6) are so-called the 'corner states' on the diagram (N_0, T) for a given irreducible representation (ωt), figure 1. The internal states of these three constructions are

$$\begin{aligned} &A(0)^{\omega - N_0} T_-^{t-\gamma} |HW\rangle \\ &A(1)^{q-T} A^+(-1)^{\omega - q} A(0)^{\alpha} |HW\rangle' \\ &A^+(-1)^{\beta} A^+(1)^{p-\beta} |vt\rangle \end{aligned} \tag{31}$$

where $q = \frac{1}{2}(\omega + t + T - N_0 - \alpha)$ and α, β, γ play the role of an additional quantum number in the three constructions to factorise the basis vectors. Let us add that in (29), (30) we have used different, but equivalent to original papers, forms of 'initial states'. In each case after the projection procedure the overcomplete base is constructed

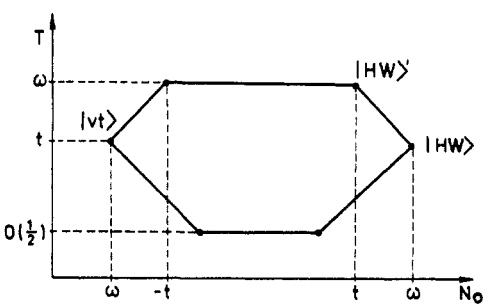

Figure 1. The schematic diagram of the 'corner states' in the (T, N_0) plane for the irreducible representation (ω, t) of the group SO(5). The 'corner states' differ for three described state constructions and are denoted by $|vt\rangle$, $|HW\rangle$ and $|HW\rangle'$.

and the problem of choosing the independent states appears in each construction. Let us make the final remark that in our construction we gave a selection rule for the states with $T > t$ while in the Smirnov-Tolstoy construction the selection rule was applied to the states with $T \leqslant t$. It also follows that our and the Smirnov-Tolstoy constructions can be considered as two complementary results of the basis problem in the group SO(5). If the multiplicity of the basis vectors with the same numbers N, T, M_T is equal to one, the three constructions give, of course, the same vectors (up to a constant factor). Moreover, in many cases with the multiplicity larger than one, the Smirnov-Tolstoy construction also gives the same vectors as in our case. We have analysed that interesting convergence and have found its explanation. If we put the projection operator (14), up to a constant factor, in the form

$$P^T_{M_T K} = P^T_{M_T T}(T_+)^{T-K} \tag{32}$$

then from (13) we get, after commuting $(T_+)^{T-K}$ to the right,

$$|(\omega t) N \beta T M_T\rangle = \sum_i \frac{P^T_{M_T T} A^+(0)^{T-t-p+2\beta-2i} A^+(-1)^{t-T+p-\beta+i} A^+(1)^{p-\beta+i}|vt\rangle}{(T-t-p+2\beta-2i)!(t-T+p-\beta+i)!\,i!\,2^i}. \tag{33}$$

In a similar way we can express the vectors of the Smirnov-Tolstoy basis. At first, let us write the exact form of the highest weight state

$$|HW\rangle' = A^+(0)^{2t} A^+(1)^{\omega-t}|vt\rangle. \tag{34}$$

If we, by using (31), apply the annihilation operators $A(0)$ and $A(1)$ to (34) then, with the projection operator $P^T_{M_T T}$, we get the Smirnov-Tolstoy vectors

$$|(\omega t) N T M_T \alpha\rangle = \sum_n \frac{P^T_{M_T T} A^+(0)^{2t-\alpha-2n} A^+(-1)^{w-q+n} A^+(1)^{T-t+w-q+n}|vt\rangle}{(2t-\alpha-2n)!(T-t+w-q+n)!\,n!\,2^n}. \tag{35}$$

The two transformed bases (33) and (35) look similar and there are special cases when the expansion coefficients in both constructions differ by a constant factor. In such a case the basis vectors are equivalent.

5. Examples of applications

We consider two problems which can be treated with the help of the constructed basis. The first problem is the problem of the fourth commuting operator in the quasi-spin physical basis. Such an operator has been constructed a long time ago (Flowers and Szpikowski 1965) in the form

$$B = \{A^+(0)^2 - 2A^+(1)A^+(-1)\}\{A(0)^2 - 2A(1)A(-1)\}. \tag{36}$$

The operator B is an annihilation and creation operator of four nucleons coupled to the total $J = 0$ and $T = 0$. It commutes with the physical operators T^2, T_0 and N. It was very soon recognised, unfortunately, that the operator B has non-rational eigenvalues in the physical basis and, hence, it cannot be interpreted as a number operator of α-like nuclear clusters. However, the diagonalisation of the operator B means at the same time the orthogonalisation of the basis. The B diagonalisation can, however, be done only non-numerically using the matrix elements of the operator B in our basis. The results of the group algebraic calculation are the following:

$$\begin{aligned} B|(\omega t) N \beta T M_T\rangle &= B_{\beta\beta}|(\omega t) N \beta T M_T\rangle + B_{\beta\beta+1}|(\omega t) N, \beta+1, T M_T\rangle \\ &\quad + B_{\beta,\beta-1}|(\omega t) N, \beta-1, T, M_T\rangle \end{aligned} \tag{37}$$

where $B_{\beta\beta'}$ are the matrix elements of the operator B given by the formulae

$$B_{\beta\beta} = (2\beta+1)(p-\beta)(\omega-t-p+\beta+1)[2(w+t-\beta)+3] + \tfrac{1}{4}(T-K)(T-K-1)(T+K+1)(T+K+2)$$

$$B_{\beta,\beta+1} = -\frac{(p-\beta)}{2(\beta+1)}(\omega-t-p+\beta+1)\sqrt{(T+K)(T+K-1)(T-K+1)(T-K+2)}$$

$$B_{\beta,\beta-1} = \tfrac{1}{2}\beta(1-2\beta)\sqrt{(T-K)(T-K-1)(T+K+1)(T+K+2)}. \tag{38}$$

One should remember that for $T > t$, to the right-hand side of equation (37) can enter the vectors outside of the chosen basis (13), (17), (23). Such vectors should first be transformed to the basis vectors and then the matrix elements calculated.

The diagonalisation of the operator B is, at the same time the orthogonalisation of the basis. Then the basis vectors are uniquely labelled by the quantum numbers T, M_T, N and by the eigenvalues of the operator B. Although, generally speaking, the eigenvalues of the B operator are non-rational numbers, they have much in common with number of α clusters and with the four-particle correlations (Szpikowski and Trajdos 1984).

The second problem in which we can exploit our basis construction is the problem of the pairing interaction with different strengths of the neutron–neutron, proton–proton and neutron–proton interactions. Hence, let us consider the problem of the diagonalisation of the pairing Hamiltonian

$$H = -\sum_{M_T} G_{M_T} A^+(M_T) A(M_T) \tag{39}$$

where, in general, the strength G_{M_T} depends on the pair isospin components M_T. If $G_{M_T} = G$ and it does not depend on M_T, then (Flowers and Szpikowski 1964) the Hamiltonian (39) is expressed by the second-order Casimir operator $C^{(2)}$, of the group SO(5) and by the commuting operators T^2 and N_0, namely

$$H = -G[C^{(2)} - \tfrac{1}{2}(T^2 - N_0^2 + 3N_0)]. \tag{40}$$

In such a case the pairing Hamiltonian is diagonal in the physical basis of the group SO(5) and the energy reads (Flowers and Szpikowski 1964)

$$E = -\tfrac{1}{4}G[(N-v)(2j+4-\tfrac{1}{2}N-\tfrac{1}{2}v) - 2T(T+1) + 2t(t+1)]. \tag{41}$$

The energy in this case does not depend either on β or M_T. In the general case (39) the Hamiltonian is deformed in the isospin space and hence, it depends as well on β as on M_T.

By the discussion of the general, pairing Hamiltonian (39) we wish to reach two goals.

(i) In the model calculation we want to recognise how the pairing energy depends on the strength G_{M_T}.

(ii) If we assume that $G_{M_T} = G =$ constant then the complicated numerical results should give the very simple result (41).

Such a comparison provides an indirect confirmation of the correct but rather complicated matrix element calculation and a confirmation of the properly constructed basis.

Let us consider one of the simple cases in which, however, the parameter β takes on at least two different values. In our example we consider $N=6$ nucleons with seniority $v=2$ and reduced isotopic spin $t=1$ in the $j=\frac{7}{2}$ level. The irreducible representation of the group SO(5) for that case is $(\omega, t)=(3, 1)$. The allowed β and T values for $N=6$ and $N=4$ are given in table 1 (the last case also enters the $N=6$ calculations). The underlined T values show the linearly independent states. Let us consider first the case of $M_T=1$. The matrix elements of (39) are calculated in the four-dimensional basis (table 1)

$$\begin{aligned}
|1_1\rangle &= |(3,1)N=6;\ \beta=1;\ T=1;\ M_T=1\rangle = P^1_{11}A^+(-1)A^+(1)|2,1\rangle \\
|1_2\rangle &= |(3,1)N=6;\ \beta=2;\ T=1;\ M_T=1\rangle = P^1_{11}A^+(-1)^2|2,1\rangle \\
|2\rangle &= |(3,1)N=6;\ \beta=1;\ T=2;\ M_T=1\rangle = P^2_{11}A^+(-1)A^+(1)|2,1\rangle \\
|3\rangle &= |(3,1)N=6;\ \beta=0;\ T=3;\ M_T=1\rangle = P^3_{13}A^+(1)^2|2,1\rangle
\end{aligned} \tag{42}$$

where $|2,1\rangle = |v=2,\ t=1\rangle$.

Let us consider, as an example of the calculation, the term

$$A^+(1)A(1)|1_1\rangle. \tag{43}$$

At first we get

$$A(1)P^1_{11} = \sum_{T,m} \frac{3(-1)^{m+1}}{2T+1}(111-1|T0)(111m|T,1+m)P^T_{0,1+m}A(-m). \tag{44}$$

The next part of the calculation gives

$$\begin{aligned}
A(1)A^+(-1)A^+(1)|2,1\rangle &= 2A^+(-1)|2,1\rangle \\
A(0)A^+(-1)A^+(1)|2,1\rangle &= -\sqrt{\tfrac{1}{2}}\,T_-A^+(1)|2,1\rangle \\
A(-1)A^+(-1)A^+(1)|2,1\rangle &= 4A^+(1)|2,1\rangle.
\end{aligned} \tag{45}$$

Hence, by (44), (45) we obtain

$$\begin{aligned}
A(1)|1_1\rangle = {} & 2P^0_{00}A^+(-1)|2,1\rangle + P^1_{00}A^+(-1)|2,1\rangle \\
& + \sqrt{\tfrac{3}{2}}\,P^2_{02}A^+(1)|2,1\rangle + \tfrac{1}{5}P^2_{00}A^+(-1)|2,1\rangle.
\end{aligned} \tag{46}$$

However, the last term on the right-hand side does not belong to the linearly independent basis (table 1), because

$$P^2_{00}A^+(-1)|2,1\rangle = \frac{1}{\sqrt{6}}P^2_{02}A^+(1)|2,1\rangle. \tag{47}$$

Table 1. The allowed values of β and T for the irreducible representation $(\omega, t)=(3,1)$ of the group SO(5). The underlined values of T are those for the linearly independent states.

N	β	T
6	0	3
	1	$\underline{1}$ $\underline{2}$ 3
	2	$\underline{1}$ 2 3
4	0	$\underline{2}$
	1	$\underline{0}$ $\underline{1}$ 2

Hence, we get

$$A(1)|1_1\rangle = 2P^0_{00}A^+(-1)|2,1\rangle + P^1_{00}A^+(-1)|2,1\rangle + \frac{16}{5\sqrt{6}}P^2_{02}A^+(1)|2,1\rangle. \tag{48}$$

By the same method of calculation we finally get the result of (43)

$$A^+(1)A(1)|1_1\rangle = \tfrac{17}{10}|1_1\rangle + \tfrac{1}{2}|1_2\rangle - \tfrac{1}{10}|2\rangle + \frac{16}{5\sqrt{15}}|3\rangle. \tag{49}$$

As a result we obtain the following matrix elements of the Hamiltonian (39) for $M_T = 1$

$$H_{11} = -1.7G_1 - 2.1G_0 - 3.2G_{-1} \qquad H_{31} = -\tfrac{1}{6}G_1 + \tfrac{5}{6}G_0 - \tfrac{2}{3}G_{-1}$$

$$H_{12} = -0.5G_1 + 0.5G_{-1} \qquad H_{32} = 0.5G_1 - 0.5G_0$$

$$H_{13} = 0.1G_1 + 3.1G_0 - 3.2G_{-1} \qquad H_{33} = -2.5G_1 - \tfrac{11}{6}G_0 - \tfrac{2}{3}G_{-1}$$

$$H_{14} = (-3.2G_1 + 6.4G_0 - 3.2G_{-1})/\sqrt{15} \qquad H_{34} = (2G_1 - \tfrac{4}{3}G_0 - \tfrac{2}{3}G_{-1})/\sqrt{15}$$

$$H_{21} = -0.2G_1 + 1.4G_0 - 1.2G_{-1} \qquad H_{41} = (-2G_1 + 4G_0 - 2G_{-1})/\sqrt{15}$$

$$H_{22} = -5G_1 - 2G_0 \qquad H_{42} = (6G_1 - 4G_0 - 2G_{-1})/\sqrt{15}$$

$$H_{23} = 6.6G_1 - 5.4G_0 - 1.2G_{-1} \qquad H_{43} = 0$$

$$H_{24} = (-1.2G_1 + 2.4G_0 - 1.2G_{-1})/\sqrt{15} \qquad H_{44} = -\tfrac{4}{5}G_1 - \tfrac{16}{15}G_0 - \tfrac{2}{15}G_{-1}.$$

By similar calculations we also consider the case with $M_T = 0$. The results are given in figure 2 where we plot the energy E in units G_0 against $x = G_1 = G_{-1}$ (also in G_0 units). The energies are relative to the energy of the ground state $E_0 = 0$.

Let us make the following comments.

(i) For $x = 1$ we get $G_1 = G_{-1} = G_0 \equiv G = \text{constant}$ and hence the formula (41) should be applied. From table 1 we get three different values of an isospin $T = 1$; 2; 3, and for these levels we get from (41) $E(T=1) = -7$; $E(T=2) = -5$; $E(T=3) = -2$ or relatively to the ground state $E = 0$; 2 and 5 exactly as from numerical calculations (figure 1 for $x = 1$).

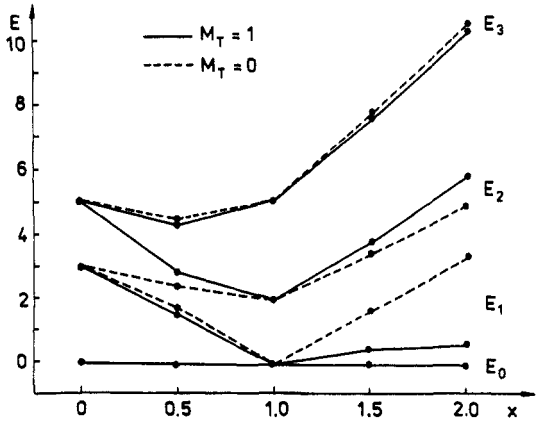

Figure 2. The energy E (in G_0 units) against the strength $x = G_1/G_0 = G_{-1}/G_0$ ($G_1 = G_{-1}$) of the pairing Hamiltonian (39) and for the irreducible representation $(\omega, t) = (3, 1)$ of the group SO(5) with $j = \frac{7}{2}$; $N = 6$; $v = 2$. The excited energies ($E_1 E_2 E_3$) are taken respectively to the ground state energy $E_0 = 0$. For $x = 1$ there is a charge-independent Hamiltonian (40) and two $T = 1$ levels are degenerated ($E_0 = E_1$).

(ii) The maximum spreading of energy levels is lower if x goes from 0 to 0.5. At first sight it looks strange because for the larger strength x, ΔE should be larger. However, detailed calculations show that the spreading depends both on diagonal and non-diagonal matrix elements of (39). Diagonal matrix elements are in fact proportional to the strength x; however non-diagonal matrix elements go to zero for $x \to 1$. Hence, opposing contributions enter the detailed calculations with the results of figure 2.

(iii) The first excited state is extremely low for $x > 1$ with $M_T = 1$. The low excited energy for that case can be explained by the large deformation in the quasi-spin and isospin abstract space with the axial symmetry ($M_T = 1$). That is not true for $x < 1$ or for $M_T = 0$.

(iv) For $x \neq 1$, two levels with $T = 1$ split in energy and, moreover, the eigenstates are not the states of a fixed T; there is not, in that case, the charge-independent Hamiltonian.

6. Concluding remarks

We have constructed a new analytical but non-orthogonal basis for the irreducible representations of the group SO(5). The basis is of a physical nature, i.e. the commuting operators which provide the quantum numbers to factorise states are isospin operators T^2 and T_0, the nucleon number operator N and the fourth β number properly defined to get the complete but not overcomplete basis. The fourth operator B can be also used to permit numerical orthogonalisation of the basis. The matrix elements of the four-particle B operator are the ingredients for the orthogonalisation procedure.

We have also discussed the known constructions of the IR basis and we have compared those constructions with the basis constructed in this paper. We have shown that all three constructions could be treated within the same projection procedure applied, however, to different initial and 'corner' states.

Finally, we have applied the constructed basis to analyse the pairing charge-dependent Hamiltonian for a system of neutrons and protons. Several interesting conclusions (section 5) follow from the detailed calculations based on the matrix elements of the Hamiltonian and on its numerical diagonalisation in the constructed new basis.

Acknowledgments

We would like to thank Dr A Góźdź for very helpful discussions.

The first author is also very grateful to Professor R Leonardi for a kind invitation and to the Group of Istituto Nazionale di Fisica Nucleare of Trento for a grant during the stay in Trento.

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